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STATIC NONLINEARITY HANDLING USING BEST LINEAR APPROXIMATION: AN INTRODUCTION

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Abstract: The engineers and scientists want mathematical models of the observed system for understanding, design and control. Most of these systems are nonlinear. There is not a unique solution because of the many different types of nonlinear systems with different behaviors and so the modeling is very involved and universally usable design tools are not available. For these reasons the nonlinear systems are often approximated with linear systems, because this theory is user friendly and well understood. In this paper we will discuss the best linear approximation in least squares sense.

Keywords: Weak nonlinearities, System identification, Best linear approximation, Frequency response function, Frequency domain approach, Detecting nonlinearities, Multisine excitation

1. Introduction

The aim of this paper is to give a short introduction to analysis of NonLinear (NL) systems and to present one of the possible ways to handle static, weakly nonlinear time invariant systems using a simple method called Best Linear Approximation (BLA), as an extension of Least Squares (LS) estimation. The definitions and the assumptions will be given with illustrative examples to make the understanding very clear.

2. Several types of nonlinearities

To get a basic definition of linearity, let us refer to Linear Time Invariant (LTI) systems. A system is linear when the principle of superposition is satisfied. If the input is denoted by u and output is denoted by y then superposition means:

$$y = f\left((a+b)u\right) = af(u) + bf(u) = (a+b)f(u), \quad a,b \in \Re, \tag{1}$$

If the system does not satisfy the assumptions above, the behavior of the observed system (denoted by f) is nonlinear. There many classes of nonlinearities depending on their behavior can be distinguished. For instance:

2.1. Static/dynamics nonlinear systems

If the steady state response $y(t)$ of the nonlinear system f to the input $u(t)$ is:

$$y(t) = f\left(u(t)\right), \quad t > 0, \tag{2}$$

then it is static nonlinear otherwise dynamic nonlinear.

2.2. PISPO/ non PISP behavior

If the steady state response of the nonlinear system to a periodic input is a periodic signal with the same period as the input has (Period In, Same Period Out, PISPO) i.e. period time T:

$$y(t) = f\left(u(t)\right) = f\left(u(t+kT)\right) = y(t+kT), \quad k \in \aleph, \tag{3}$$

then the system has PISPO behavior.

Remark that all the static processes have PISPO behavior, but the opposite is not true. There are some dynamical systems, which have PISPO behavior, e.g. Wiener-Hammerstein systems [1].

2.3. In-Band/Out-of-Band nonlinearities

If the effect of nonlinear behavior is limited to the excited frequency band then it is called as In-Band distortion, otherwise it is Out-of-Band distortion.

2.4. Even/odd nonlinearities

The system contains even/odd nonlinear distortion when energy has been transferred from any excited frequency to some even/odd harmonic and it has been added to the linear response, this is further illustrated in Section 4.2. Of course the system can have both kinds of nonlinear distortions in the same time.

2.5. Coherent/non-coherent contributions

The nonlinear system has coherent contribution, if the phases at the output of the excited system deterministically follow the changing of the phases at the input.

In case of nonlinear distortions it is proven [2] that the odd nonlinear systems can be coherent or non-coherent and the even nonlinear systems can only be non-coherent.

2.6. Weakly nonlinear system/ nonlinear system

A nonlinear system is weak, if the nonlinear part can be represented by the following equation:

$$y = cu^n, \quad n \le 3, \quad c = \text{const.} \tag{4}$$

The fundamental difference between a Static NonLinear (STNL) system and a linear time invariant system is that a nonlinear system transfers energy from one (excited) frequency to other (excited or non-excited) frequencies, which is impossible in case of LTI system. This is illustrated by the following example.

2.7. An Example

Let us consider an ideal noiseless quadratic system whose input signal is an ideal cosine wave with amplitude A=1 and with angular frequency ω (see *Fig. 1*).

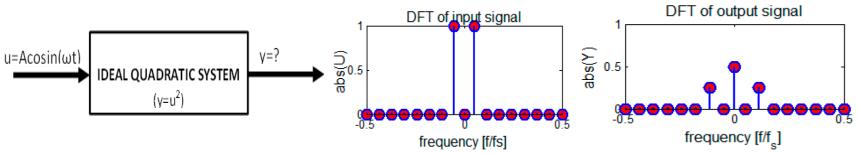

Fig. 1. An example of weakly nonlinear system: a quadratic system. On the left side the system structure is shown, on the right side the input and output spectra are presented

The system output can be calculated very easily with help of Euler's formula:

$$y(t) = u^2(t) = \left(A\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2} \right) \right)^2$$
$$= \frac{A^2}{4}\left(e^{2j\omega t} + e^{-2j\omega t} + 2 \right) = \frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega t). \tag{5}$$

In the above example a quadratic non-linear, static system is considered. It contains even nonlinearity because energy has been transferred from the odd frequencies (in this case from 1ω,-1ω) to the even harmonics (to 2ω,-2ω,0).

3. Basic questions related to the topic system identification

Several important questions must be addressed prior to carrying out any system identification procedure.

3.1. Which kinds of nonlinearities are allowed?

Because there are many types of nonlinear classes there is no unique and universal paradigm for modeling nonlinear systems. For that reason it is needed to make some restrictions and assumptions for the observed system to be able to model it. In this paper it is assumed that the observed system has only weak nonlinearities. In this question it is also implicitly hidden that which kind of describing method will be used. The possibilities are for instance:

- Volterra models (it is an extension of Taylor series with memory effect) [3];
- Block oriented plant models [4] (which is used in BLA);
- Nonlinear subspace model [5]

3.2. Which convergence criterion will be used?

Convergence criteria are important, when understanding the limiting nature of a function or variable. Basically there are two kinds of convergence criteria:

- uniform convergence [6]: typically, a session (in *Fig. 2* it is in the rectangular domain) is observed and estimated within an error bound, mostly with Taylor and Volterra series;
- pointwise convergence [6]: an arbitrary neighborhood around the observed points is chosen. In the system identification the frequently used concept is the convergence in mean square sense. This convergence criterion will be also used to define the Best Linear Approximation, see in Section 6.

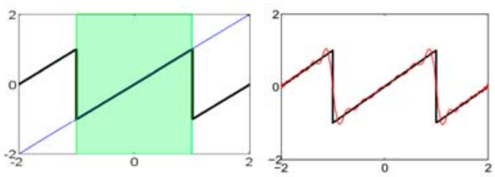

Fig. 2. An example: on the left side is the uniform convergence is used for the proposed domain, and on the right side in pointwise convergence is shown for the whole interval

3.3. Which excitation signal will be applied?

The answer of a nonlinear system depends strongly on:

i) The nature of the excitation signal:

- amplitude distribution (power density function, pdf) e.g. uniform, Gaussian distribution, or sinusoidal, for further details see Section 5.1;

- power spectral density (PSD) e.g. low pass, high pass etc.;
- excitation level: small, medium, large, see for further details see Section 4.2.

ii) The type of excitation:

- periodicity: repeated, non-repeated, see *Fig. 3*;
- randomness: random or self-constructed (e.g. full, odd, even multisine), see Section 4;
- length.

For the reasons above, the design of experiment is needed. The main requirements for experiment design: reduce the influence of the noise (reach the highest Signal-to-Noise Ratio, SNR), robust to nonlinear distortions, robust to misuse and user friendly. Let us give an example. In case of linear plant model the higher the excitation amplitude, the better SNR value will be given, which (in general) leads to the smallest observation error. But in case of nonlinear process the effects of nonlinear behavior will be increasing with the amplitude. In the next Section one powerful technique is discussed.

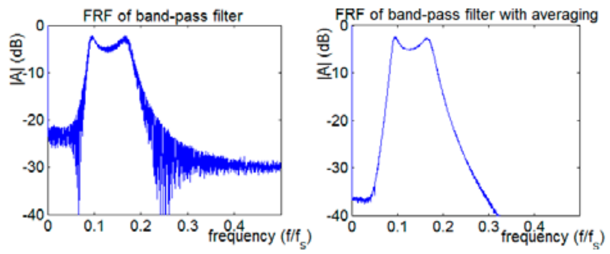

Fig. 3. Effect of averaging in presence of disturbing noise. On the left side the FRF of an band-pass filter is shown without averaging, on the right side is the same band-pass filter with averaging (with 16 iterations)

4. Multisine excitation and detection of nonlinearities

4.1. Multisine excitation

To avoid any spectrum leakage, to reach full nonparametric characterization of the noise and to be able to detect nonlinearities, a periodic signal is needed. Many users prefer noise excitations, because it looks much simpler (see *Fig. 4.*), but in this case nonlinearities are not identifiable. The best that satisfies the assumptions in Section 3.2 is the multisine excitation (see *Fig. 4*), which looks like noise but it is not a noise.

The random phase (uniformly distributed) multisine is a sum of harmonically related sinusoids as shown in the following

$$u_{ms}(t) = \sum_{k=1}^{F} A_k \cos(k\omega_0 t + \varphi_k), \quad \varphi_k \sim \cup(0,2\pi), \tag{6}$$

where ω_0, is the fundamental angular frequency and F is the highest harmonic component. The amplitude distribution of a random phase multisine approaches a Gaussian distribution as the number of harmonics tend to infinity.

If the multisine contains all/only odd or even harmonics then it is full/odd or even multisine.

Remark, the multisine excitation is not equivalent to swept (stepped, chirp sine) [7].

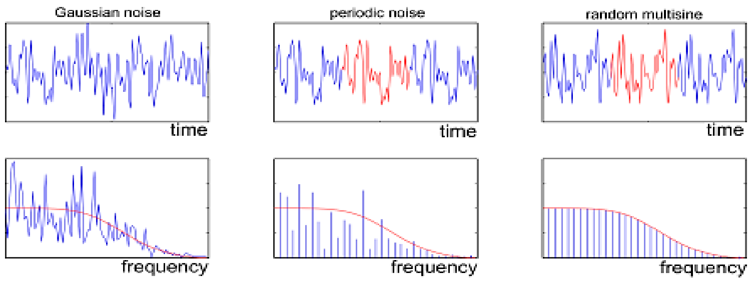

Fig. 4. Different excitation signals in time and in frequency domain

4.2. Detecting nonlinearities

When small excitation level is used, the effects of nonlinearities can be hidden in the noise. If the excitation level is higher, the nonlinearities will appear. If a full-band excitation is used, then the details of nonlinear behavior are not separable from the linear part. With an excitation set of only even harmonics, the odd nonlinearities are not detectable. The solution to detect both kinds (even/odd) of nonlinearities is to use an excitation set only with odd harmonics, e.g. odd random phase multisine. In order to examine the odd frequencies, it is needed to skip several odd harmonics in the multisine. The experiences show that the odd, random phase multisine with randomly skipped harmonics is the best what can be used. The *Fig. 5* shows an example, how the system response consists of the linear and nonlinear part. In the *Fig. 6* an experiment with an NL Low-pass filter (SilverBox) is shown: the higher the amplitude, the higher effect of nonlinear behavior can be caught.

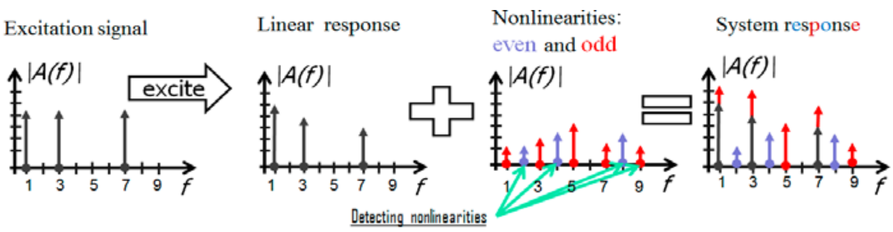

Fig. 5. System response originating from linear and nonlinear part of excited system

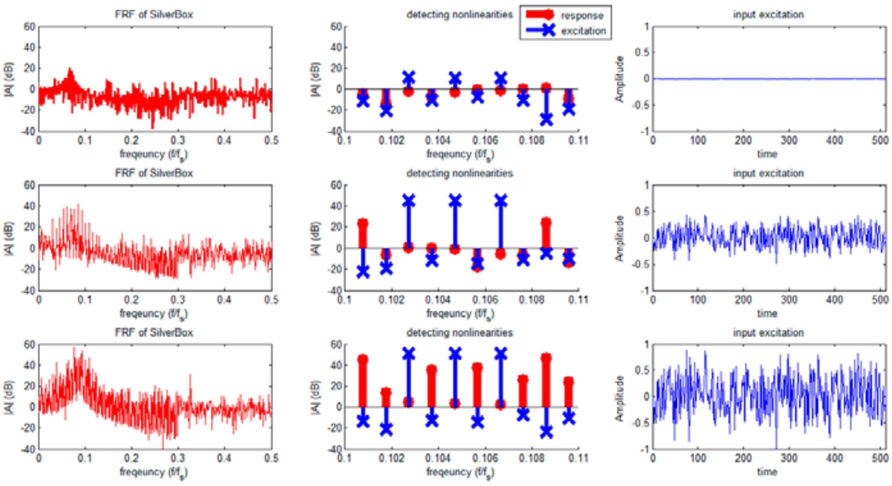

Fig. 6. Experiment with a nonlinear low-pass filter (SilverBox) with different level of excitation (small, medium, high). The left column shows the FRF of system, the middle column shows the excited (normalized) frequencies between 0.1 and 0.11 f/fs. On the left column the input excitation in time domain is shown

5. Weakly nonlinear systems with best linear approximation approach

The BLA of a nonlinear system is an approach of modeling what minimizes the mean square error between the true output of nonlinear system and the output of the linear model (represented by the impulse response function $g(t)$). For instance this definition is in time domain (see also *Fig. 7*):

$$\min\left\{\left\|\tilde{y}(t)-g(t)*\tilde{u}(t)\right\|_F^2\right\}=\min\{S\} \tag{7}$$

where * is the convolution product [5], and $\tilde{y}(t)=y(t)-E\{y(t)\}$, $\tilde{u}(t)=u(t)-E\{u(t)\}$ for further details, see Section 5.2.

5.1. Theoretical structure and the basic assumptions

In *Fig. 7* the theoretical structure of BLA is considered with the following components [2]:

G_{Linear} represents the true underlying linear system if it exists (otherwise it is zero). Its response is independent of the power spectrum of excitation signal (because of the linearity).

G_{Bias} denotes the systematic nonlinear contribution whose spectrum is smooth [1], [2], [8]. Its value is fixed in the same excitation set.

Thus G_{BLA} (as an addition of G_{Linear} and G_{Bias}) has also smooth Frequency Response Function (FRF) [7], which is same for the same excitation [4]. Later it is discussed, how it is possible to decrease the influence of nonlinear parts.

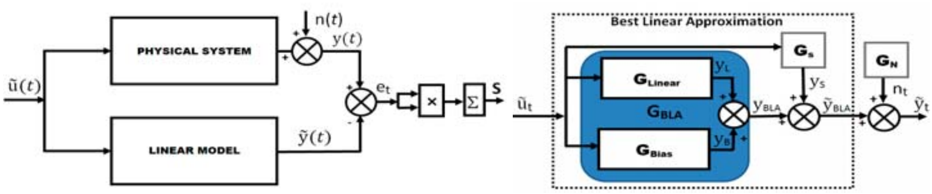

Fig. 7. The theoretical structure of the best linear approximation

G_S has been modeled as a 'half-stochastic' nonlinear noise generator with the very important properties:

- has circular complex normal distribution with zero mean (it means stochastic, for different excitation sets belong to different values, but for the same set belongs to same value);
- same smooth spectrum for same excitation;
- even and odd nonlinearities can be contributed.

The properties of the nonlinear G_{Bias} and G_S part in the BLA model are depending on

- the energy level of each excited frequency;
- the probability density function of the input (see *Fig. 8*);
- even and odd nonlinear distortions of the system;
- type of the disturbing noise and its impact point.

Noise n_t represents the 'regular' noise at the system output. In this article it is assumed that it has zero mean normal distribution with variance - σ^2, i.e. $n_t \sim N(0, \sigma^2)$.

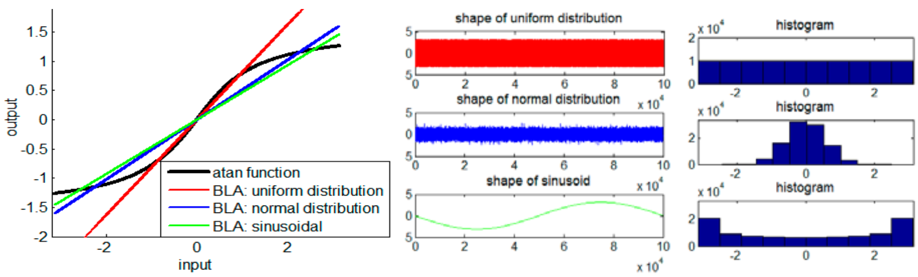

Fig. 8. Estimation of atan function between -π and π with different pdf of input signal

Thus, assuming that there is no measurement or round-off error, the complex frequency function is for BLA with leaving the transient term:

$$\hat{G}_{BLA}(j\omega) = G_{Bias}(j\omega) + G_{Linear}(j\omega) + G_S(j\omega) \\ = G_{BLA}(j\omega) + G_S(j\omega), \tag{8}$$

and with the transfer function of G_N for the whole observed system:

$$\hat{G}_{observed}(j\omega) = \hat{G}_{BLA}(j\omega) + G_N(j\omega), \tag{9}$$

5.2. Importance of removing the mean values

In case of using the best linear approximation the DC (mean) values of the input and output signals must be removed. This is the best linearization (in RMS sense) around the operating point, so this is one of the important differences between the regular LS and the BLA. *Fig. 9a* shows an example for the simulated, noiseless STNL system of characteristic $y(t)=u^2(t)$ with uniformly distributed input between 1 and 2. With the mean values $\bar{y}, \bar{u}$ the estimated least squares regression line i.e. Linear Approximation (LA) and BLA outputs are

$$\hat{y}_{LSR,t} = u_t \hat{g}_{LSR} = u_t \frac{Y^T U}{U^T U}, \tag{10}$$

$$\hat{y}_{BLA,t} = u_t \hat{g}_{BLA} = (u_t - \bar{u}) \frac{(Y - [e_1, e_2, \cdots, e_N])^T (U - [e_1, e_2, \cdots, e_N]\bar{u})}{(U - [e_1, e_2, \cdots, e_N]\bar{u})^T (U - [e_1, e_2, \cdots, e_N]\bar{u})} + \bar{y}. \tag{11}$$

5.3. Cascading

The BLA of the cascade of two nonlinear systems is not equal to the cascade of BLAs of each nonlinear system separately: an example is shown in *Fig. 9b*.

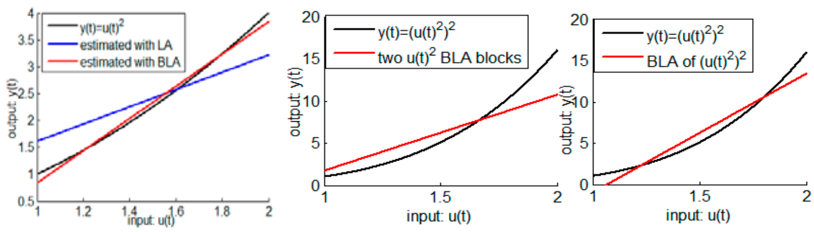

Fig. 9. a) estimation of $y(t)=u(t)^2$ with LA and with BLA; b) Estimates of $y=(u(t)^2)^2$ with BLA as cascaded $y=u(t)^2$ two blocks and one $y=(u(t)^2)^2$ block

6. Effect of Gaussian excitation and noise

In this Section the effect of Gaussian-like signals is studied (due to limited length, only a summary is given to this topic).

For Gaussian noise (and for every kind of odd random phase multisine [7]) excitations, the BLA of static nonlinearity is static and independent of the coloring of the power spectrum and the nonlinear G_{Bias} term is independent on even nonlinearities.

However, the BLA of a static nonlinearity is dynamic for non-white, non-normally distributed inputs. In both (Gaussian and non-Gaussian) cases the variance of the BLA measurement depends on the coloring of the power spectrum. Filtered white non-Gaussian noise behaves asymptotically (as the length of the impulse response of the filter tends to infinity) as a Gaussian one.

If the length of the random phase multisine is sufficiently large [2], then the systematic nonlinear contribution G_{Bias} does not depend on harmonics of the spectrum, as long as the equivalent power frequency band remains the same. However, the stochastic nonlinear noise part of the BLA, G_S strongly depends on the harmonic content in all cases.

7. The effects of two dimensional averaging

In this article it is assumed that the observed system is static weakly nonlinear disturbed only with Gaussian noise. The system is only excited by Gaussian-like signals e.g. random phase multisine [7]. It is very important that G_{BLA}, G_S and G_N satisfy the assumption given in the Section 5.1.

In order to identify a process (system) there has to average over p periods belonging to the same excitation set and over the m different realizations (with different excitation set). For the observed system with the notations above, in frequency domain:

$$\hat{G}_{observed}^{[m][p]}(j\omega) = \frac{\hat{Y}_{observed}^{[m][p]}(j\omega)}{U_{exc}^{[m]}(j\omega)} = G_{BLA}(j\omega) + G_S^{[m]}(j\omega) + G_N^{[m][p]}(j\omega). \tag{12}$$

First, let us get the effect the averaging over the p periods in the same excitation set with leaving the transient term. If p is sufficiently large then (originated from the law of large numbers [9] and the distribution of n_t) [10] the expected value of the observed system:

$$\begin{aligned} E\left\{G_{observed}^{[m]}(j\omega)\right\} &= \frac{1}{p}\sum_{i=1}^{p} G_{observed}^{[m],i}(j\omega) = \frac{1}{p}\sum_{i=1}^{p}\begin{pmatrix} G_{BLA}(j\omega) \\ + G_S^{[m]}(j\omega) + G_N^{[m],i}(j\omega)\end{pmatrix} \\ &= G_{BLA}(j\omega) + G_S^{[m]}(j\omega) + 0 = G_{BLA}(j\omega) + G_S^{[m]}(j\omega) \end{aligned} \tag{13}$$

and so the variance:

$$Var\left\{\hat{G}^{[m]}_{observed}(j\omega)\right\} = \sum_{i=1}^{p} \frac{\left|\hat{G}^{[m],i}_{observed}(j\omega) - \hat{G}^{[m]}_{observed}(j\omega)\right|}{p(p-1)}. \tag{14}$$

Second, let us average over the *m* realizations. If *m* is sufficiently large, then:

$$\begin{aligned} E\left\{\hat{G}_{observed}(j\omega)\right\} &= \frac{1}{m}\sum_{j=1}^{m} \hat{G}^{j}_{observed}(j\omega) = \frac{1}{m}\sum_{j=1}^{m}\left(\hat{G}_{BLA}(j\omega) + G^{j}_{S}(j\omega)\right) \\ &= \hat{G}_{BLA}(j\omega) + 0 = G_{BLA}(j\omega) \end{aligned} \tag{15}$$

and thus the variance is:

$$\begin{aligned} Var\left\{\hat{G}^{[m]}_{observed}(j\omega)\right\} &= \sum_{i=1}^{p} \frac{\left|\hat{G}^{[m],i}_{observed}(j\omega) - \hat{G}_{observed}(j\omega)\right|^2}{m(m-1)} \\ &= Var\left\{\hat{G}_{N}(j\omega)\right\} + Var\left\{\hat{G}_{S}(j\omega)\right\}, \end{aligned} \tag{16}$$

where the variances of the regular and nonlinear noises are (originated from the distribution properties):

$$Var\{G_N(j\omega)\} = \frac{Var\left\{G^{[m][p]}_{N}(j\omega)\right\}}{pm}, \quad Var\{G_S(j\omega)\} = \frac{Var\left\{G^{[m]}_{S}(j\omega)\right\}}{m}. \tag{17}$$

This means that the influence of the noise and non-linear contribution can be decreased with this special two-dimensional averaging, as a final result the $\hat{G}_{obs}(j\omega)$ (BLA estimator) will be provided. In [2], [10], [11] it has been shown that the above statements are only true with the condition and assumptions declared in Section 5, when the number of periods is greater or equal than two ($p \geq 2$) and the different realizations are more than six ($m > 6$). An overview about the steps of averaging used for BLA estimator can be found in *Fig. 10*.

8. Conclusion

The proposed best linear approximation estimator is very useful for weakly static nonlinearities because it originated from linear theory of plant modeling. It is simple to use without advanced knowledge in statistics and good fitting is possible for several kinds of non-linear systems (e.g. Wiener-Hammerstein systems).

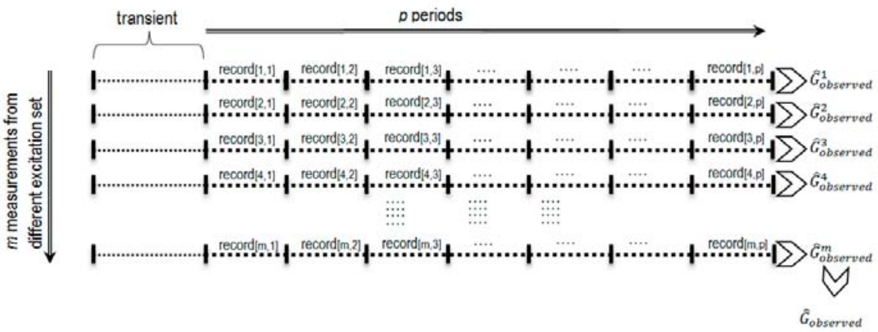

Fig. 10. Evaluation of $\hat{G}_{obs}(j\omega)$ with the help of two dimensional averaging

On the other hand it is only restricted usability guaranteed for this special class of nonlinearities. Furthermore, when the excitation set is not Gaussian-like, then $\hat{G}_{observed}(j\omega)$ can be more complicated and dynamical.

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